Chapter 6: Set theory

Def'n: A set is a well-defined collection of objects.

Examples:

$\mathbb{Z}$: set of all integers

$\mathbb{N}$ or $\mathbb{Z}^{\geq}$: set of all non-negative integers
$0, 1, 2, 3, \ldots$

$\mathbb{Z}^+$: set of all positive integers: $1, 2, 3, \ldots$

$\mathbb{Q}$: set of all rational numbers

$\mathbb{R}$: set of all real numbers

$\mathbb{C}$: set of all complex numbers.

Notation: $x \in E$

element or member

"belongs to"

set

To denote the proposition

$$\neg(x \in E)$$

we write $x \notin E$.

does not belong to

Ways of specifying a set:

(1) Simply listing them.

examples: $A = \{1, 7, 3/4\}$

$= \{7, 1, 3/4\}$

Order does not matter

$= \{7, 7, 1, 1, 1, 3/4\}$

repeating elements does not change A

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

(2) Using conditional definitions

examples:

- $B = \{ n \in \mathbb{Z} \mid 0 < n < 6 \}$

(this is a different way of writing $B = \{1, 2, 3, 4, 5\}$.

(3) Using a constructive definition.

examples

$$C = \{ n^2 \mid n \in \mathbb{Z} \}$$

$$= \{ 0, 1, 4, 9, \ldots \}$$

$$\mathbb{Q} = \{ a/b \mid a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \}.$$

So here $(q \in \mathbb{Q}) \Leftrightarrow [(q = a/b) \wedge (a \in \mathbb{Z}) \wedge (b \in \mathbb{Z}) \wedge (b \neq 0)]$.

Definition: The empty set, denoted by ϕ, is the unique set with no elements at all.

Example: $\phi = \{x \in \mathbb{R} \mid x^2 + x + 1 = 0\}$

Definition: Given two sets A and B, we say A is a subset of B, denoted

$$\boxed{A \subseteq B \quad (\text{or } B \supseteq A)}$$

when every element of A is also an element of B. That is

$$\boxed{(x \in A) \Rightarrow (x \in B)}$$

Remark: In the book $A \subset B$ means that A is a proper subset of B, i.e., $(A \subseteq B) \wedge A \neq B)$

I prefer using $A \subset B$ to mean $A \subseteq B$

Note that:

(i) $(x \in A) \Leftrightarrow (\{x\} \subseteq A)$

(ii) $(A = B) \Leftrightarrow [(A \subseteq B) \wedge (B \subseteq A)]$

(iii) $(P(a) \Rightarrow Q(a) \text{ for all } a \in A)$
$$\Leftrightarrow \left[\{a \in A \mid P(a)\} \subseteq \{a \in A \mid Q(a)\}\right]$$

(iv) $(A \subseteq B) \wedge (B \subseteq C) \Rightarrow (A \subseteq C)$

(v) $\emptyset \in A$ for all sets A.

6.2 Operations on sets

Definition: The intersection of two sets A and B is the set of elements belonging to both A and B

$$\boxed{A \cap B = \{x \mid x \in A \text{ and } x \in B\}.}$$

Definition: We say A and B are disjoint if $A \cap B = \emptyset$.

Definition: The union of two sets A and B is the set of elements belonging to A or B

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Definition: The difference of two sets A and B is the set of elements belonging to A but not B

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}.$$

Remark: Often $A \setminus B$ is used in place of $B \setminus A$.

Remark: In general $A \setminus B$ and $B \setminus A$ don't have to be equal.

example: Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$,

then $A \setminus B = \{3\}$

and $B \setminus A = \emptyset$.

Note: $A \cap A = A \cup A = A$,

$A \cap \emptyset = \emptyset$,

$A \cup \emptyset = A$,

$A \setminus A = \emptyset$,

$A \setminus \emptyset = A$.

Proposition: Given two sets A and B, the sets $A \cap B$, $B \setminus A$, $A \setminus B$ are pairwise disjoint. Moreover $A \cup B = (A \cap B) \cup (A \setminus B) \cup (B \setminus A)$.

Proof: Consider the truth table

$x \in A$	$x \in B$	$x \in A \cap B$	$x \in A \setminus B$	$x \in B \setminus A$
T	T	T	F	F
T	F	F	T	F
F	T	F	F	T
F	F	F	F	T

and note that each row in the last 3 columns has at most one T. This proves that

$A \cap B$, $A \setminus B$, $B \setminus A$ are pairwise disjoint.

Now consider the truth table:

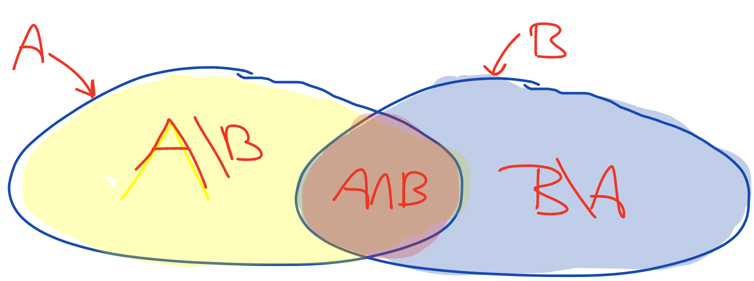
$x \in A$	$x \in B$	$x \in (A \cap B) \cup (A \setminus B) \cup (B \setminus A)$	$x \in (A \cup B)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

which proves that

$$A \cup B = (A \cap B) \cup (A \setminus B) \cup (B \setminus A).$$

$\blacksquare$

Illustration via Venn diagrams:



Definition

The power set of a set X, denoted $\mathcal{P}(X)$, is the set of all subsets of X.

$$A \in \mathcal{P}(X) \iff A \subseteq X$$

Examples

If $X = \{1\}$, $\mathcal{P}(X) = \{\phi, \{1\}\}$.

singletons

If $X = \{a, b\}$, $\mathcal{P}(X) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

If $X = \{1,2,3\}$, $\mathcal{P}(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Definition

Given a fixed universal set U and a set $A \in \mathcal{P}(U)$, the complement of A, denoted by A^c, is

$$A^c = U \setminus A = \{x \in U \mid x \notin A\}.$$

Theorem: Let U be a universal set and let $A, B, C \in \mathcal{P}(U)$, then:

(i) $A \cup (B \cup C) = (A \cup B) \cup C$.
$A \cap (B \cap C) = (A \cap B) \cap C$. (associativity)

(ii) $A \cup B = B \cup A$ and $A \cap B = B \cap A$. (commutativity)

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. (distributivity).

(iv) $(A \cup B)^c = A^c \cap B^c$.
$(A \cap B)^c = A^c \cup B^c$. (De Morgan's laws)

(v) $A \cup A^c = U$, $A \cap A^c = \emptyset$. (complementation)

(vi) $(A^c)^c = A$. (double complementation)

We will prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, the rest are an exercise.

Proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(I) We will first prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:

If $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in (B \cup C)$. Moreover, as $x \in (B \cup C)$ then $x \in B$ or $x \in C$. We consider both cases; if $x \in B$, $x \in A \cap B$ (since $x \in A$). Similarly if $x \in C$, $x \in A \cap C$.

Thus $x \in (A \cap B) \cup (A \cap C)$, and $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

(II) proof that $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$:

Suppose that $x \in (A \cap B) \cup (A \cap C)$, then $x \in A \cap B$ or $x \in A \cap C$. In the first case $x \in A$ and $x \in B$, which implies $x \in B \cup C$. In the second case ($x \in A \cap C$), $x \in A$ and $x \in C$, hence $x \in B \cup C$.

Thus in both case $x \in A$ and $x \in B \cup C$ so $x \in A \cap (B \cup C)$.

Exercise: Prove that
$$(A \cup B) \cap (C \cup D) = (A \cap C) \cup (B \cap C) \cup (A \cap D) \cup (B \cap D).$$

Chapter 7: Quantifiers

7.1 Universal Statements

Sometimes a predicate $P(a)$ (where a takes values in the set A) is true for some values of a and false for others.

If $P(a)$ is true for all values of a in A we say that $P(a)$ is a universal statement.

We use the notation

$$\forall a \in A, P(a)$$

"For every/for each/for all"

to mean $\{a \in A \mid P(a)\} = A$.

Example: The statement

$$\forall a \in \mathbb{R}\setminus\{0\}, \quad a^2 > 0$$

is equivalent to the statement

$$\{a \in \mathbb{R}\setminus\{0\} \mid a^2 > 0\} = \mathbb{R}\setminus\{0\},$$

and also equivalent to the statement

$$a \in \mathbb{R}\setminus\{0\} \Rightarrow a^2 > 0.$$

7.2 Existential Statements

Suppose we want to negate the (false) universal statement

$$\forall x \in \mathbb{R}, (x > 3 \Rightarrow x > 5).$$

In particular, the above statement is false bec. there is an x $(x = 4)$ for which $x > 3$ and $x \not> 5$

Def'n: there exists an element a in A for which $P(a)$ holds.

The notation $\exists a \in A, P(a)$ means

there exists/for some/for at least one

$$\{a \in A \mid P(a)\} \neq \emptyset.$$

Examples:

$\exists n \in \mathbb{Z}, n^2 = 9$ is a true statement

$\exists n \in \mathbb{Z}, n + 1 = 4.5$ is false

$\exists x \in \mathbb{R}, x + 1 = 4.5$ is true.

Proving Statements involving quantifiers

(1) Statements of the form: $\boxed{\forall a \in A, P(a)}$

example: $\forall x \in \mathbb{R}, x^2 \geq 0$

we already proved this by cases, the idea is to cover all possibilities.

One way to do this is to prove the equiv. statement: $a \in A \Rightarrow P(a)$.

(2) Statements of the form: $\boxed{\exists a \in A, P(a)}$

Here, we just have to find one (or more) $a \in A$ s.t. $P(a)$ holds.

Example: $\exists x \in \mathbb{R}, \; x^2 - 2x + 1 = 0.$

Proof: Take $x = 1$, then $x^2 - 2x + 1 = 1^2 - 2 + 1 = 0$.

(3) Statements involving both quantifiers.

Usually these statements are of the form:

$$\forall a \in A, \; \exists b \in B, \text{ such that } P(a,b).$$

Example: $\forall$ even integer n, $\exists q \in \mathbb{Z}$, $n = 2q$.

In words: For every even number n, we can find some (other) number q, such that $n = 2q$.

(Of course, this is the def'n of even numbers).

Disproving statements involving quantifiers:

To disprove a statement of the form $\boxed{\forall a \in A, P(a)}$ we just have to find a counterexample.

That is, we prove the statement

$$\underbrace{\exists a \in A, \text{ not } P(a)}_{\text{This is the negation of } (\forall a \in A, P(a)).}$$

Example: To disprove $(\forall x \in \mathbb{R}, x^2 > 0)$

we just have to note that $0 \in \mathbb{R}, 0^2 \not> 0$.

To disprove a statement of the form $\boxed{\exists a \in A, P(a)}$

we again have to prove its negation, which is

$$\underbrace{\forall a \in A, \neg P(a)} = \neg(\exists a \in A, P(a)).$$

Statements involving more than one free variable.

- $\forall a \in A, \forall b \in B, P(a,b)$.

  Example: $\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z} \setminus \{0\}, \frac{a}{b} \in \mathbb{Q}$.

  (This is the def'n of $\mathbb{Q}$).

- $\exists a \in A, \exists b \in B, P(a,b)$

  Example: $\exists m \in \{1,2,3\}, \exists n \in \{2,3,4\}$,

  $$\frac{m}{n} = \frac{6}{8}.$$

  Proof: $3 \in \{1,2,3\}$, $4 \in \{2,3,4\}$

  and $\frac{3}{4} = \frac{6}{8}$. $\square$

- $\forall a \in A, \exists b \in B, P(a,b)$

  Example: $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}, m < n$.

  Proof: $\forall m \in \mathbb{Z}$, $m+1 \in \mathbb{Z}$ and $m < m+1$.

- $\exists b \in B, \forall a \in A, P(a,b)$.

  Example: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy = y$.

  Proof: Take $x = 1 \in \mathbb{R}$, then $\forall y \in \mathbb{R}, 1 \cdot y = y$.

- $\forall b \in B, \exists a \in A, P(a,b)$.

  Example: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y = x^2$.

The Cartesian Product of two sets

Defn: The Cartesian product of two sets X and Y, denoted by $X \times Y$ is the set of all ordered pairs (x,y) where $x \in X, y \in Y$.

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\}.$$

Remark: $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

Example: $A=\{1,2\}$, $B=\{a,b,c\}$

$A\times B=\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)\}$,

$B\times A=\{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$.

notice: $A\times B\neq B\times A$

Remark: When $X=Y$, $X\times Y=X\times X\underset{\text{notation.}}{=}X^2$.

Example: $\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$ is the usual 2-dimensional space.

Cartesian Products and Predicates $P(a,b)$

Example: Let $x\in\mathbb{R}$, $y\in\mathbb{R}$ and consider the predicate $P(x,y): y=2x$.

and note $\{(x,y)\in\mathbb{R}^2 \mid P(x,y)\}=\{(x,y)\in\mathbb{R}^2 \mid y=2x\}$
$\subset\mathbb{R}^2$ is the line of equation $y=2x$.

Proposition: For all sets A, B, C, D

(i) $A\times(B\cup C) = (A\times B)\cup(A\times C)$

(ii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$

(iii) $(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D)$

(iv) $(A\times B)\cup(C\times D) \subseteq (A\cup C)\times(B\cup D)$.

Proof of (i):

$$
\begin{aligned}
(x,y)\in A\times(B\cup C) &\iff [(x\in A)\wedge(y\in B\cup C)] \\
&\iff (x\in A)\wedge((y\in B)\vee(y\in C)) \\
&\iff [(x\in A)\wedge(y\in B)]\vee[(x\in A)\wedge(y\in C)] \\
&\iff (x\in A\times B)\vee(x\in A\times C) \\
&\iff (x,y)\in(A\times B)\cup(A\times C).
\end{aligned}
$$

$\blacksquare$

Exercise: • Do the remaining proofs.

• Find an example of A, B, C, D, so that

$$(A\times B)\cup(C\times D) \neq (A\cup C)\times(B\cup D).$$